

# Coupled uncertainty provided by a multifractal random walker

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The aim here is to study the concept of pairing multifractality between time series possessing non-Gaussian distributions. The increasing number of rare events creates "criticality". We show how the pairing between two series is affected by rare events, which we call "coupled criticality". A method is proposed for studying the coupled criticality born out of the interaction between two series, using the bivariate multifractal random walk (BiMRW). This method allows studying dependence of the coupled criticality on the criticality of each individual system. This approach is applied to data sets of gold & oil markets, and inflation & unemployment.

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## I. INTRODUCTION

The concept of coupling features emerge when two or more systems are taken under consideration [1–3]. Every one has heard of macroscopic coupling in nature, e.g. in waves, interfaces, in modern life such as social, economical & political issues [4–6], coupling phenomena in condensed matter physics [7, 8] and in the context of neuroscience [9] are just some examples in this regard. The most important aspect of coupling which motivated this work is assessing the criticality or in other words the uncertainty born out of coupled systems. Such coupling exists between companies, stock markets, surfaces & interfaces, stochastic fields and, etc.

The multifractal formalism provides almost adequate tools for studying the scaling relations and properties of objects possessing a fractal geometry and/or generalized multifractal exponents [10–14]. Once, the infinitely divisible cascades [15–17] in hydrodynamic turbulence was confirmed to be statistically scale-invariant, one important motivation for multifractal formulation to become prominent was established [18, 19]. Multifractal models have been implemented in various fields of sciences ranging from biology, geology, social to finance, e.g. Earth & Solar winds [20–22], foreign exchange rates [23], stock index [24, 25], human heartbeat fluctuations [26, 27], well-log data [28] seismic time series [29–32], and sol-gel transitions [33–35].

A breakthrough in the applications of multifractal models took place when the relation between turbulence and finance in the context of multiplicative random cascades was established [17, 23, 36]. Note that their manifestation was based on their results that the velocity increment fluctuations in turbulence and financial returns are proportional through a stochastic factor. This factor only depends on the scale ratio of the processes. In a further study, by developing this theory, it has been shown that a continuous random walk model can possess multi-

fractal properties due to the existence of a correlation in the logarithm of the stochastic variances [13, 17]. It is worth noting that a strong log-normal deviation from the normal case leads to a robust state of multifractality or in other words a critical state in the underlying system. This is due to the fact that, since the distribution function has a fat tail, the occurrence of low frequent events gets to be more probable compared to their corresponding in the normal distribution. This is why and how the criticality enters the system.

In the present study we show how the individual uncertainty or criticality of each system affects their coupled criticality. To this end, we study the criticality in coupled systems by implementing the bivariate multifractal random walk method. However, it could be instructive to read the applications of this method in other disciplines [37–41].

This paper is organized as follows: In section II method for analysis will be explained. Data description and implementation of method are explained in section III. Section IV is devoted to summary and conclusions.

## II. MODEL AND ANALYSIS

Consider a stochastic process represented by  $x(t)$ , which may be a function of both space and time. More complementary explanations can be found in [13, 42, 43]. Here we assume that time is a dynamical parameter, therefore  $x(t)$  is only taken to be time dependant. The increment of  $x(t)$  at time lag  $\ell$  is defined as  $\Delta_\ell x(t) \equiv x(t + \ell) - x(t)$ . According to the cascading approach, the increment of fluctuations  $\Delta_\ell x(t) \equiv x(t + \ell) - x(t)$ , at scales  $\ell$  and  $\eta \times \ell$  satisfy the following relation

$$\Delta_{(\eta \times \ell)} x(t) = \mathcal{W}_\eta \Delta_\ell x(t), \quad \forall \ell, \eta > 0, \quad (1)$$

where  $\mathcal{W}_\eta$  is a stochastic variable [15, 44]. We assume that the cascading process starts from a large scale,  $L$ ,

where by starting the iterating process would eventually tend to small scales ( $\ell < L$ ). For a multiplicative cascading process which starts from a large scale,  $L$ , tending to small scales,  $\ell$ , implementation of the multifractal random walk approach enables us to rewrite the increment of fluctuation as  $\Delta_\ell x(t) \equiv \xi_\ell(t)e^{\omega_\ell(t)}$ , in which  $\xi_\ell(t)$  and  $\omega_\ell(t)$  are independent of each other and have Gaussian distributions with zero means. The corresponding variances are denoted by  $\sigma^2(\ell)$  and  $\lambda^2(\ell)$  for  $\xi_\ell(t)$  and  $\omega_\ell(t)$ , respectively [13]. In this approach, a non-Gaussian probability density function (PDF) with fat tails is expressed by [15]

$$\mathcal{P}_\ell(\Delta_\ell x) = \int G_\ell(\ln \sigma(\ell)) \frac{1}{\sigma(\ell)} F_\ell\left(\frac{\Delta_\ell x}{\sigma(\ell)}\right) d \ln \sigma(\ell), \quad (2)$$

where we have

$$G_\ell(\ln \sigma(\ell)) = \frac{1}{\sqrt{2\pi}\lambda(\ell)} \exp\left(-\frac{\ln^2 \sigma(\ell)}{2\lambda^2(\ell)}\right), \quad (3)$$

$$F_\ell\left(\frac{\Delta_\ell x}{\sigma(\ell)}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\Delta_\ell x^2}{2\sigma^2(\ell)}\right). \quad (4)$$

Simply, one can show that in the limit where  $\lambda^2(\ell)$  tends to zero,  $\mathcal{P}_\ell(\Delta_\ell x)$  converges to a Gaussian function. Increasing the parameter  $\lambda^2(\ell)$  quantifies the efficiency of the non-Gaussianity, where the tail of the profile starts to fatten. Therefore, a large value of  $\lambda^2(\ell)$  indicates a high probability of finding large fluctuations in a data set. This statement is a reminiscence of criticality in the system as pointed out in [24]. It is worth noting the long-range correlation and/or non-Gaussianity are the sources of multifractality nature of a stochastic field. In this paper we concentrate on the shape of the probability density function which acts a source for multifractality characterized by  $\lambda^2(\ell)$  [45, 46].

Since it is possible that two neighbouring sites interact with each other, the verification of their correlation would be of interest. This brings up the idea that the coupled behaviour may correspond to the behaviour of each individual system together with their cross correlation. In this line Muzy et al. considered the cross correlation between processes by generalizing the multifractal random walk approach based on the log-normal cascade model [47]. This generalization introduces the multivariate multifractal model which describes the scale invariance of the joint statistical properties. Suppose  $\mathbf{x}(t) \equiv \{x_1(t), x_2(t)\}$  is represented as a bivariate process, the bivariate version of *BiMRW* for the increment  $\Delta_\ell \mathbf{x}(t) = \mathbf{x}(t + \ell) - \mathbf{x}(t)$  reads as [47]

$$\Delta_\ell \mathbf{x}(t) = \left( \xi_\ell^{(1)}(t) e^{\omega_\ell^{(1)}(t)}, \xi_\ell^{(2)}(t) e^{\omega_\ell^{(2)}(t)} \right), \quad (5)$$

where for each increment at the time lag  $\ell$  there exists a separate  $\xi$  and  $\omega$ . Note that the bivariate processes  $[\xi_\ell^{(1)}, \xi_\ell^{(2)}]$  and  $[\omega_\ell^{(1)}, \omega_\ell^{(2)}]$  are independent of one another, both having a joint Gaussian distributions with a zero mean. The covariance matrices  $\Sigma_\ell$  and  $\Lambda_\ell$  respectively become

$$\begin{aligned} \Sigma_\ell &\equiv \begin{pmatrix} \Sigma_\ell^{(11)} & \Sigma_\ell^{(12)} \\ \Sigma_\ell^{(21)} & \Sigma_\ell^{(22)} \end{pmatrix} \\ &= \begin{pmatrix} \langle \xi_\ell^{(1)}(t) \xi_\ell^{(1)}(t) \rangle & \langle \xi_\ell^{(1)}(t) \xi_\ell^{(2)}(t) \rangle \\ \langle \xi_\ell^{(2)}(t) \xi_\ell^{(1)}(t) \rangle & \langle \xi_\ell^{(2)}(t) \xi_\ell^{(2)}(t) \rangle \end{pmatrix}, \\ \Lambda_\ell &\equiv \begin{pmatrix} \Lambda_\ell^{(11)} & \Lambda_\ell^{(12)} \\ \Lambda_\ell^{(21)} & \Lambda_\ell^{(22)} \end{pmatrix} \\ &= \begin{pmatrix} \langle \omega_\ell^{(1)}(t) \omega_\ell^{(1)}(t) \rangle & \langle \omega_\ell^{(1)}(t) \omega_\ell^{(2)}(t) \rangle \\ \langle \omega_\ell^{(2)}(t) \omega_\ell^{(1)}(t) \rangle & \langle \omega_\ell^{(2)}(t) \omega_\ell^{(2)}(t) \rangle \end{pmatrix}. \end{aligned} \quad (6)$$

The four diagonal elements of the presented matrixes, namely  $\Sigma_\ell^{(11)} \equiv \sigma_1^2(\ell)$ ,  $\Sigma_\ell^{(22)} \equiv \sigma_2^2(\ell)$ ,  $\Lambda_\ell^{(11)} \equiv \lambda_1^2(\ell)$ , and  $\Lambda_\ell^{(22)} \equiv \lambda_2^2(\ell)$  are defined for the two individual processes 1 and 2. In addition, the symmetry property of these matrices implies the following equalities;  $\Sigma_\ell^{(12)} = \Sigma_\ell^{(21)} \equiv \Sigma_\ell \sigma_1(\ell) \sigma_2(\ell)$  and  $\Lambda_\ell^{(12)} = \Lambda_\ell^{(21)} \equiv \Lambda_\ell \lambda_1(\ell) \lambda_2(\ell)$ . Usually,  $\Sigma_\ell$  is called the "*Markowitz matrix*" which shows the variance and correlation of  $\xi$ 's.  $\Lambda_\ell$  is the "*multifractal matrix*" which quantifies the non-linearity of  $\omega$ 's [47, 48]. In this framework, the shape of joint-PDF would be

$$\begin{aligned} \mathcal{P}_\ell(\Delta_\ell x_1, \Delta_\ell x_2) &= \int d(\ln \sigma_1(\ell)) \int d(\ln \sigma_2(\ell)) \\ &G_\ell(\ln \sigma_1(\ell), \ln \sigma_2(\ell)) \frac{1}{\sigma_1(\ell) \sigma_2(\ell)} F_\ell\left(\frac{\Delta_\ell x_1}{\sigma_1(\ell)}, \frac{\Delta_\ell x_2}{\sigma_2(\ell)}\right), \end{aligned} \quad (7)$$

where  $G_\ell(\ln \sigma_1(\ell), \ln \sigma_2(\ell))$  and  $F_\ell\left(\frac{\Delta_\ell x_1}{\sigma_1(\ell)}, \frac{\Delta_\ell x_2}{\sigma_2(\ell)}\right)$  are the probability density functions of the bivariate processes  $(\omega_\ell^{(1)}, \omega_\ell^{(2)})$  and  $(\xi_\ell^{(1)}, \xi_\ell^{(2)})$ , respectively. By taking into account the cross correlation between the two systems under consideration, their joint-probability density function in the integral of would be

$$G_\ell(\ln \sigma_1(\ell), \ln \sigma_2(\ell)) = \frac{1}{2\pi\lambda_1(\ell)\lambda_2(\ell)\sqrt{(1-\Lambda_\ell^2)}} \times \\ \exp\left(-\frac{1}{2(1-\Lambda_\ell^2)}\left[\left(\frac{\ln^2 \sigma_1(\ell)}{\lambda_1^2(\ell)}\right) + \left(\frac{\ln^2 \sigma_2(\ell)}{\lambda_2^2(\ell)}\right) - 2\Lambda_\ell\left(\frac{\ln \sigma_1(\ell)}{\lambda_1(\ell)}\right)\left(\frac{\ln \sigma_2(\ell)}{\lambda_2(\ell)}\right)\right]\right),$$

and

$$F_\ell\left(\frac{\Delta_\ell x_1}{\sigma_1(\ell)}, \frac{\Delta_\ell x_2}{\sigma_2(\ell)}\right) = \frac{1}{2\pi\sqrt{(1-\Sigma_\ell^2)}} \times \\ \exp\left(-\frac{1}{2(1-\Sigma_\ell^2)}\left[\left(\frac{\Delta_\ell x_1^2}{\sigma_1^2(\ell)}\right) + \left(\frac{\Delta_\ell x_2^2}{\sigma_2^2(\ell)}\right) - 2\Sigma_\ell\left(\frac{\Delta_\ell x_1}{\sigma_1(\ell)}\right)\left(\frac{\Delta_\ell x_2}{\sigma_2(\ell)}\right)\right]\right). \quad (8)$$

It turns out that if the cross correlation coefficients  $\Lambda_\ell$  and  $\Sigma_\ell$  are to be zero, consequently  $G_\ell$  and  $F_\ell$  would simply be the multiplication of the two independent processes. This result confirms that any deviation from the product of these two independent processes would lead to the coupling of the two processes. According to the covariance matrix defined in Eq. (6), the parameter  $\Lambda_\ell$  controls the strength of the joint multifractality for the two processes [47].

In order to estimate the parameter  $\Lambda_\ell$  at scale  $\ell$ , we use the Bayesian statistics [50, 51]. Consider an original data set and a theoretical model that is able to create the total of  $N$  non-Gaussian data sets with corresponding ( $N$ ) theoretical PDFs. For each point of the original PDF, there exist  $N$  equivalent points extracted from the theoretical PDFs, where due to the central limit theory their PDF is Gaussian. In this line we write the corresponding likelihood function [49],  $\mathcal{L}$ , in the form of a multivariate Gaussian function is

$$\mathcal{L}(\mathcal{P}_{\text{data}}(\mathbf{y}, \ell) | \mathcal{P}_{\text{theory}}(\mathbf{y}; \boldsymbol{\Lambda}_\ell, \boldsymbol{\Sigma}_\ell)) \quad (9) \\ = \frac{\sqrt{\text{Det}\{\mathcal{F}\}}}{(2\pi)^{N/2}} \exp\left(-\frac{\Delta^T \cdot \mathcal{F} \cdot \Delta}{2}\right)$$

where  $\Delta \equiv \mathcal{P}_{\text{data}}(\mathbf{y}, \ell) - \mathcal{P}_{\text{theory}}(\mathbf{y}; \boldsymbol{\Lambda}_\ell, \boldsymbol{\Sigma}_\ell)$  is a column vector,  $\mathcal{F}$  is the fisher information matrix which is determined according to  $\mathcal{F}^{-1} = \langle \Delta(\mathbf{y})\Delta(\mathbf{y}') \rangle$ .  $\mathbf{y}$  is an independent parameter and  $(\boldsymbol{\Lambda}_\ell, \boldsymbol{\Sigma}_\ell)$  is a model free parameter.  $\mathcal{P}_{\text{data}}(\mathbf{y}, \ell)$  is computed directly from the data sets, and  $\mathcal{P}_{\text{theory}}(\mathbf{y}; \boldsymbol{\Lambda}_\ell, \boldsymbol{\Sigma}_\ell)$  is estimated from Eq. (7). Since there is no reason to have cross correlation between  $\Delta(\mathbf{y})$ 's for different  $\mathbf{y}$ 's the fisher matrix would be diagonal (if there are non-zero cross correlation in covariance matrix then one can use proper similarity transformation to diagonalize this Hermitian matrix). The best fit for all scales which has the maximum likelihood with the original PDF, is found using the  $\chi^2$  test. This is where  $\chi^2$  obtains its global minimum

$$\chi^2(\boldsymbol{\Lambda}_\ell; \boldsymbol{\Sigma}_\ell) = \sum_{\mathbf{y}} \frac{[\mathcal{P}_{\text{data}}(\mathbf{y}, \ell) - \mathcal{P}_{\text{theory}}(\mathbf{y}; \boldsymbol{\Lambda}_\ell; \boldsymbol{\Sigma}_\ell)]^2}{\sigma_{\text{data}}^2(\mathbf{y}, \ell) + \sigma_{\text{theory}}^2(\mathbf{y}; \boldsymbol{\Lambda}_\ell; \boldsymbol{\Sigma}_\ell)}, \quad (10)$$

where  $\sigma_{\text{data}}^2(\mathbf{y}, \ell)$  and  $\sigma_{\text{theory}}^2(\mathbf{y}; \boldsymbol{\Lambda}_\ell, \boldsymbol{\Sigma}_\ell)$  are the mean standard deviation of  $\mathcal{P}_{\text{data}}(\mathbf{y})$  and  $\mathcal{P}_{\text{theory}}(\mathbf{y}; \boldsymbol{\Lambda}_\ell, \boldsymbol{\Sigma}_\ell)$ , respectively. By marginalizing over nuisance free parameter,  $\boldsymbol{\Sigma}_\ell$ , we obtain

$$\chi^2(\boldsymbol{\Lambda}_\ell) = \sum_{\mathbf{y}} \int d\boldsymbol{\Sigma}_\ell \left( \frac{[\mathcal{P}_{\text{data}}(\mathbf{y}, \ell) - \mathcal{P}_{\text{theory}}(\mathbf{y}; \boldsymbol{\Lambda}_\ell, \boldsymbol{\Sigma}_\ell)]^2}{\sigma_{\text{data}}^2(\mathbf{y}, \ell) + \sigma_{\text{theory}}^2(\mathbf{y}; \boldsymbol{\Lambda}_\ell, \boldsymbol{\Sigma}_\ell)} \right). \quad (11)$$

The best fit values for the non-Gaussian parameters  $\boldsymbol{\Lambda}_\ell$  are determined systematically by searching in the landscape of marginalized chi-square.

The cross correlation function of the processes is obtained by

$$C_\ell^{\text{joint}}(\tau) \equiv \langle [\bar{\omega}_\ell^{(1)}(i) - \langle \bar{\omega}_\ell^{(1)} \rangle][\bar{\omega}_\ell^{(2)}(i + \tau) - \langle \bar{\omega}_\ell^{(2)} \rangle] \rangle. \quad (12)$$

As stated earlier, the upper indices "1" and "2" refer to the processes (1) and (2) respectively, and  $\tau$  represents the time lag where  $\ell < \tau$ . Due to the fact that the vector process is stationary, the cross correlation is only dependent on  $\tau$ . The parameter  $\bar{\omega}_\ell^{(\diamond)}(i)$  is the local variance defined by

$$\bar{\omega}_\ell^{(\diamond)}(i) = \frac{1}{2} \ln \sigma_{(\diamond)}^2(\ell, i), \quad (13)$$

where its magnitude is

$$\sigma_{(\diamond)}^2(\ell; i) = \frac{1}{\ell} \sum_{j=1+(i-1)\ell}^{i\ell} \Delta_\ell x_\diamond^2(j). \quad (14)$$

The symbol  $(\diamond)$  should be replaced by (1) and (2) for the first and second data sets [24, 26, 27, 52].

### III. APPLICATIONS OF BIMRW

In this section, we apply our method to the reciprocal effects of oil and gold markets which have been recorded on daily basis at the time interval from 1995 to 2012 [53].

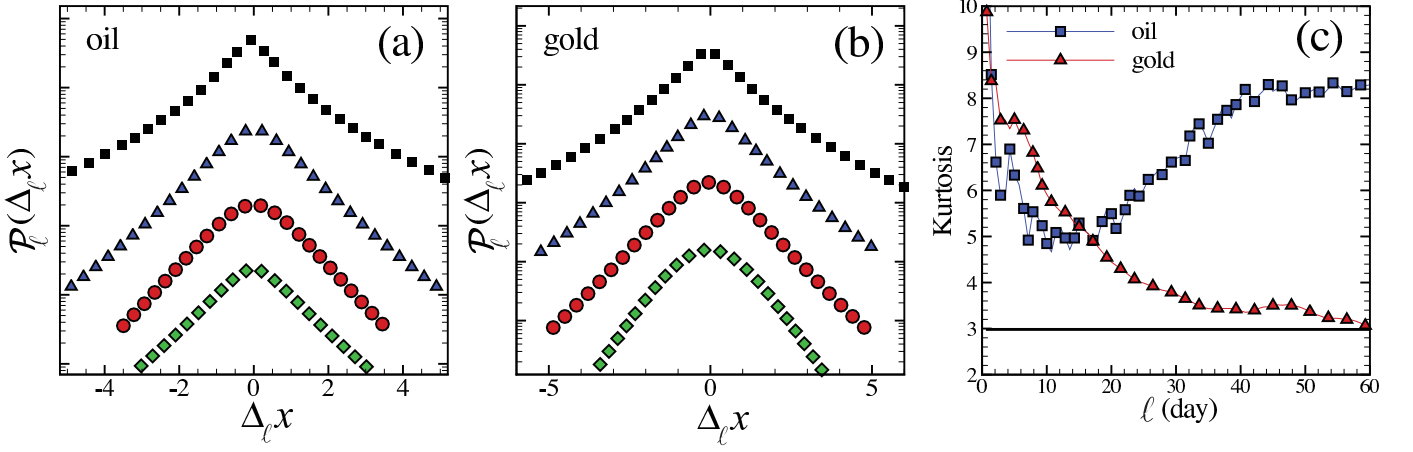


FIG. 1: Panels (a) and (b) indicate probability density function of oil and gold markets from top to bottom for daily (square symbol), weekly (delta symbol), monthly (circle symbol), and seasonally (diamond symbol) time scales. Panel (c) shows Kurtosis of the markets versus time lag,  $\ell$ .

As discussed in detail earlier in the text, in *BiMRW*, the joint multifractal parameter,  $\Lambda_\ell$ , describes the coupling of large fluctuations of two processes [47]. As a matter of fact a large value of  $\Lambda_\ell$  refers to a robust joint multifractality which results in a coupled criticality or uncertainty state in the system. The scaling parameter,  $\Lambda_\ell$ , which plays an important role in  $G_\ell(\ln \sigma_1(\ell), \ln \sigma_2(\ell))$ , can be written as [47]:

$$\Lambda_\ell = \frac{\Lambda_\ell^{(12)}}{\lambda_1(\ell)\lambda_2(\ell)} = \frac{\langle \ln \sigma_1(\ell) \ln \sigma_2(\ell) \rangle}{\lambda_1(\ell)\lambda_2(\ell)}, \quad (15)$$

where  $\langle \dots \rangle$  denotes the ensemble averaging on all windows with size  $\ell$ . It could be deduced from Eq. (15) that the scaling parameter,  $\Lambda_\ell$ , is affected by two parameters namely, the non-Gaussian parameters  $\lambda$ 's, and the cross correlation of the stochastic variances  $\langle \ln \sigma_1(\ell) \ln \sigma_2(\ell) \rangle$ . Note that since  $\Lambda_\ell$  depends on the scale  $\ell$ , the scale in which  $\Lambda_\ell$  ascends must be carefully investigated. Actually, a large  $\Lambda_\ell$  implies the emergence of coupled criticality or uncertainty. The various situations that  $\Lambda_\ell$  probably comes up are as follows:

- if two underlying systems are uncorrelated, due to independency of associated individual uncertainty which are quantified by  $\lambda$ 's, consequently the systems will not experience any coupling. Hence there is no uncertainty between them.

- in case of correlated systems, the coupled criticality can emerge when at least one of underlying systems possess a Gaussian distribution. In this situation, the  $\lambda$ 's tend to zero. Such condition is a benchmark of resonance. However, in this normal state, the weak cross correlation of large fluctuations results in decreasing  $\Lambda_\ell$ .

- if two underlying systems are correlated, but both of them are in their high criticality states, by increasing

the  $\lambda_\ell$ 's, the coupled uncertainty decreases.

Regarding these explanations we will assess how the coupled uncertainty exists and examine whether this coupled uncertainty is generally directed or not. In principle, we expect that a system associated with a more uncertainty state with large  $\lambda$ 's, would lead to the occurrence of large fluctuations causing an uncertainty state in the neighbouring system. This implies that the two systems are correlated, therefore, the coupled uncertainty becomes large. This means that  $\Lambda_\ell$  also increases. Subsequently, one can deduce that  $\Lambda_\ell$  behaves similar to the conditional uncertainty of a system that is more normal and impressed by the other system. In contrast, if the system that is in the normal state causes large fluctuations in the neighbouring system which is in a higher uncertainty state, the cross correlation of large fluctuations decreases, resulting in decreasing of  $\Lambda_\ell$ .

The corresponding stochastic parameters for further investigation are daily records of  $x(t)$  for oil and gold markets [53]. The analysis is based on the stationary increment fluctuations of the markets. According to Eq. (5), the bivariate model for underlying data sets have been determined. Probability density function based on cascading approach have been illustrated in Fig. 1. This plots confirm that a non-Gaussian behaviour at scales smaller than a month exists for both series. Gold market has a Gaussian probability density function at larger scales. This statement is proved by panel (c) of Fig. 1, where the kurtosis of the markets has been plotted. One can also deduce that the kurtosis of the markets at scales less than a month is greater than 3. For scales greater than a month, the kurtosis for the gold market reaches to 3. Thus, one would expect a Gaussian distribution for the gold market at scales much larger than a month. To get more reliable results, we have also computed non-Gaussian parameter  $\lambda_o^2(\ell)$  according to the

likelihood statistics explained in the previous section for both data sets and indicated in Fig. 2. As shown in panel (a) of Fig. 2, the value of  $\lambda_1^2(\ell)$  for the oil market is high at all scales especially at large scales, however, for the gold market the large value of  $\lambda_2^2(\ell)$  turns up at small scales. Thus, the oil market resembles a market with a continuous criticality compared to the gold market which shows criticality only at small scales. This behaviour is in agreement with Fig. 1 (a). In addition, the joint multifractal parameter,  $\Lambda_\ell$ , has been plotted versus scales  $\ell$  in Fig. 2. It could be noticed that  $\Lambda_\ell$  rises during the time interval [25–40] days. A large value of  $\Lambda_\ell$  reveals a state with a strong coupled criticality that is a confirmation for the existence of resonance in the market. This is due two following reasons; the first comes from the normal behaviour of the gold market ( $\lambda_{\text{gold}}^2 = \lambda_2^2 \rightarrow 0$ , indicated in Fig. 2 (a)). The second is because of a powerful coupling of large fluctuations of the oil and gold markets. This is due to a long-range cross correlation of the stochastic magnitudes at a monthly scale, see Fig. 2 (b). For time scales before and after the powerful coupling the scenario is different. For time scales before the strong coupling or in other words time scales smaller than a month, the joint multifractal parameter,  $\Lambda_\ell$ , shows a decreasing in the market. Indeed the existence of an individual criticality state or in other words large value of  $\lambda_\diamond^2(\ell)$ 's for each market accompanied by a weak magnitude cross correlation for a weekly scale (Fig. 2 (b)) results in the decreasing of the coupled criticality (reducing  $\Lambda_\ell$ ). On the other hand, as shown in Fig. 2 (b), at large scales or scales after the strong coupling, the oil market with large  $\lambda_1^2(\ell)$  has a short-range magnitude cross correlation with the gold market for a seasonal scale. This issue again causes a decrease of criticality in coupled markets.

A question that arises here is whether the occurrence of an uncertainty state in one system produced by the other system is a directed phenomenon or not. In order to answer this question, we must first show how the occurrence of large fluctuations and a high criticality in one system are affected by large fluctuations in the other system. To this end, we should evaluate the variance of the conditional distribution for the stochastic local magnitudes. The variance of conditional distribution of the local magnitudes is defined by

$$\text{Var}(\bar{\omega}_\ell^{(1)}|\bar{\omega}_\ell^{(2)}) \equiv \sum_i \left[ \bar{\omega}_\ell^{(1)}(i) - \langle \bar{\omega}_\ell^{(1)} \rangle \right]^2 \mathcal{P}(\bar{\omega}_\ell^{(1)}(i)|\bar{\omega}_\ell^{(2)}(i)), \quad (16)$$

where the conditional distribution function  $\mathcal{P}(\bar{\omega}_\ell^{(1)}(i)|\bar{\omega}_\ell^{(2)}(i))$ , can be read as

$$\mathcal{P}(\bar{\omega}_\ell^{(1)}(i)|\bar{\omega}_\ell^{(2)}(i)) = \frac{\mathcal{P}(\bar{\omega}_\ell^{(1)}(i), \bar{\omega}_\ell^{(2)}(i))}{\mathcal{P}(\bar{\omega}_\ell^{(2)}(i))}, \quad (17)$$

where  $\mathcal{P}(\bar{\omega}_\ell^{(1)}(i), \bar{\omega}_\ell^{(2)}(i))$ , is the joint distribution function of the stochastic local magnitude,  $\bar{\omega}_\ell^{(\diamond)}$ , for the two underlying processes. The large conditional variance is

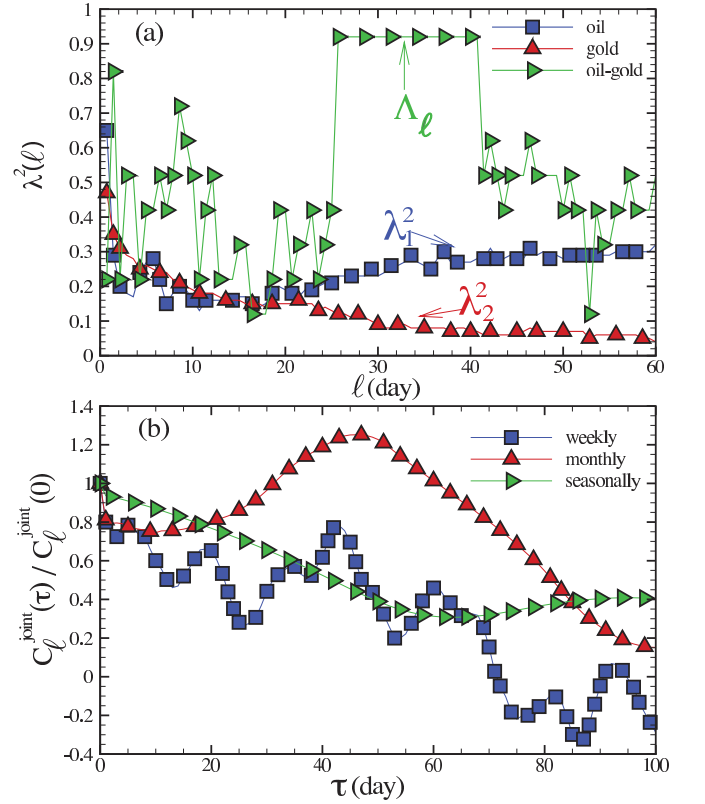


FIG. 2: The scale dependence of the non-Gaussian parameter,  $\lambda^2(\ell)$ , for oil and gold markets recorded daily at the time interval between 1995 to 2012 has been shown in panel (a). The coupling multifractal parameter  $\Lambda_\ell$  for the joint markets has been also plotted versus scale,  $\ell$ . Panel (b) illustrates the magnitude of cross correlation function of oil and gold markets,  $C_\ell^{\text{joint}}(\tau)/C_\ell^{\text{joint}}(0)$  versus time lag,  $\tau$ , for weekly (square symbol), monthly (triangle symbol) and seasonally (right triangle symbol) scales.

due to the occurrence of large fluctuations in a system through the fluctuations in the other system. Nonetheless, a system with higher criticality condition (large  $\lambda_\diamond^2(\ell)$ ) would result in the occurrence of large fluctuations accompanied by a high conditional criticality in the other system. This phenomenon is due to a strong cross correlation of large fluctuations between the two systems. Now, we introduce the concept of directed coupled criticality. Strictly speaking this concept means that if a coupling between two systems is considered as non-balanced, or e.g. in social life as a one sided relation such as a one sided love, one of the people (systems) would be in an uncertainty (criticality) state with the other while this is not true (or an option) the other way round. In order to investigate this directed coupled uncertainty, the conditional cross correlation function of processes (1) and (2) at time scale  $\ell$  versus time lag  $\tau$  ( $\ell < \tau$ ) must be defined.

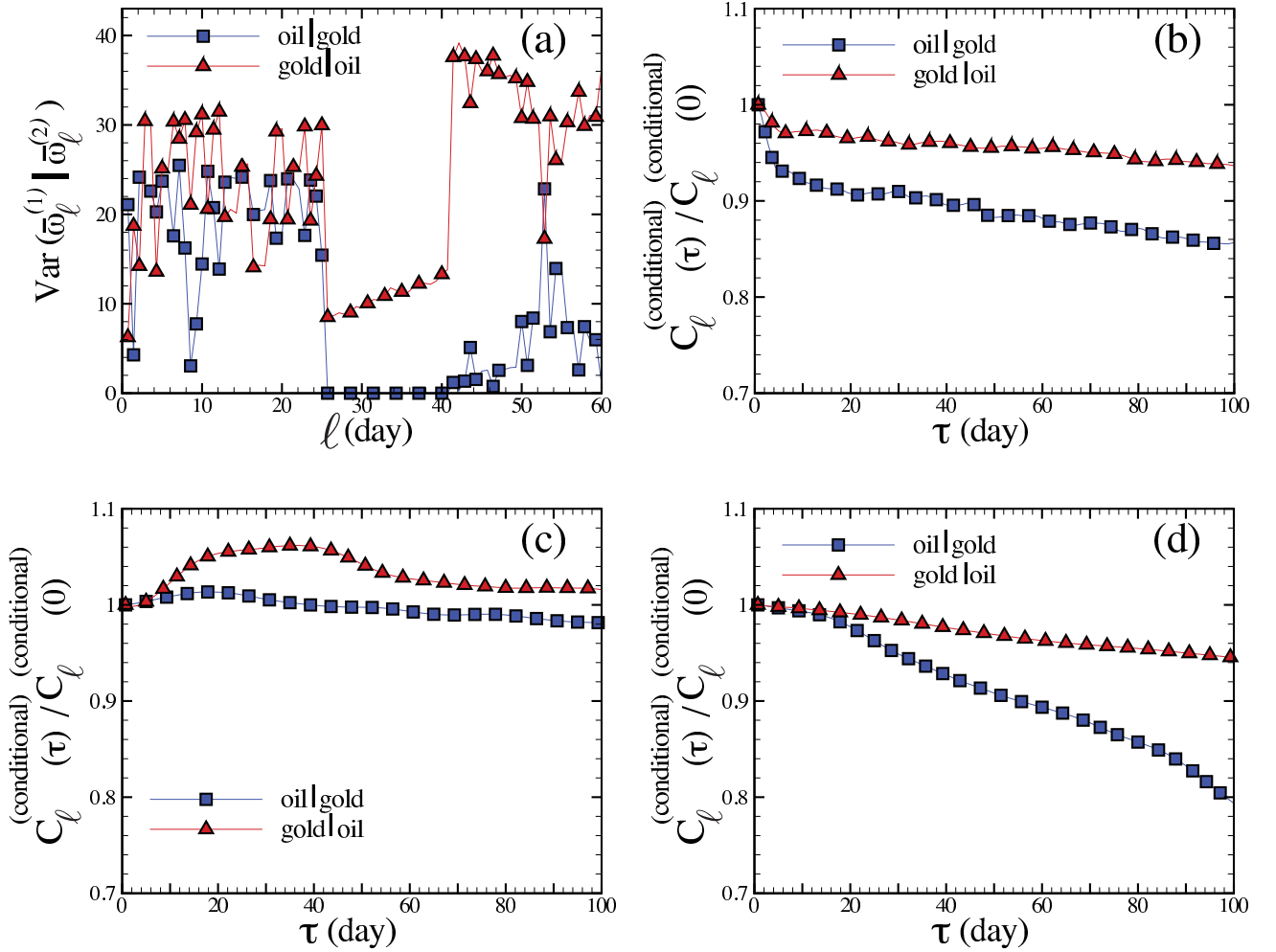


FIG. 3: Panel (a): Variance of the conditional distribution of the local magnitudes versus scale,  $\ell$ , for oil and gold markets. Panels (b), (c) and (d) indicate the conditional magnitude cross correlation function of oil and gold markets,  $C_\ell^{\text{conditional}}(\tau)/C_\ell^{\text{conditional}}(0)$ , as a function of time lag  $\tau$ , for weekly, monthly and seasonally scales, respectively.

Therefore, the conditional cross correlation is:

$$C_\ell^{(\text{conditional})}(\tau) \equiv \sum_i \left[ \bar{\omega}_\ell^{(1)}(i) - \langle \bar{\omega}_\ell^{(1)} \rangle \right] \left[ \bar{\omega}_\ell^{(2)}(i + \tau) - \langle \bar{\omega}_\ell^{(2)} \rangle \right] \times \mathcal{P} \left( \bar{\omega}_\ell^{(1)}(i) | \bar{\omega}_\ell^{(2)}(i) \right). \quad (18)$$

Fig. 3(a) shows the conditional variance of the stochastic magnitudes of the markets corresponding to the definition expressed by Eq. (16). The conditional magnitude variance of the gold market is more than the oil market at almost all scales. This means that the oil market with a large uncertainty or criticality produces large fluctuations in the gold market. This originates the fact that the conditional cross correlation of the gold market is long-range at all scales, see panels (b,c,d) of Fig. 3. The comparison of Fig. 3 (a) and Fig. 2 (a) shows that the behaviour of the conditional variance of the oil market,  $\text{Var}(\bar{\omega}_\ell^{(\text{oil})} | \bar{\omega}_\ell^{(\text{gold})})$  is almost opposite to behaviour of  $\Lambda_\ell$ .

It must point out that, the variance of the oil market which is considered to have a high uncertainty, is caused by the gold market (which is a normal market), would result in a weak cross correlation between the two markets. Hence, the coupled uncertainty decreases, meaning that  $\Lambda_\ell$  decreases. It is readily noticed by Fig. 3 (a) that at the resonant region, about a one month time-scale, the conditional variance of the oil market tends to zero. This expresses that the gold market which is normal at this time scale ( $\lambda_{\text{gold}}^2(\ell) \rightarrow 0$ ) does not produce noticeable changes in the oil market, thus, the conditional variance of the oil market takes its minimum value.

It is instructive to provide another application in compliance with the discussions of the present study. Due to the different reports, it is known that inflation and unemployment possess an inverse proportionality with each other. In other words, decreasing of one of them causes to increase other [54]. This has been a trick for governments to control one by the other one. As of the context of the present study the interest is to look at inflation and

unemployment from the view point of coupled criticality.

By extracting data from the U.S. Inflation Calculator and the U.S. Bureau of Labor Statistics [55] we study the time series of inflation and unemployment rates and compare their non-Gaussian parameters,  $\lambda_{\diamond}^2(\ell)$ . It is shown in Fig. 4 that for small scales the non-Gaussianity of unemployment rates is greater than for inflation rates. But towards larger scales it is the inflation rate that possesses a greater non-Gaussian parameter. But most interesting of all is their joint non-Gaussian parameter ( $\Lambda_{\ell}$ ) which indicates that in small scales and large scales their behaviour is proportional to each other ( $\Lambda_{\ell} > 0$ ), while at the intermediate range, namely from month 6 to month 26 their behaviour is inversely proportional to each other ( $\Lambda_{\ell} < 0$ ). This provides an insight on the joint uncertainty or coupled criticality represented by  $\Lambda_{\ell}$  which is non-zero, regarding the long-range cross correlation of large fluctuations between two systems.

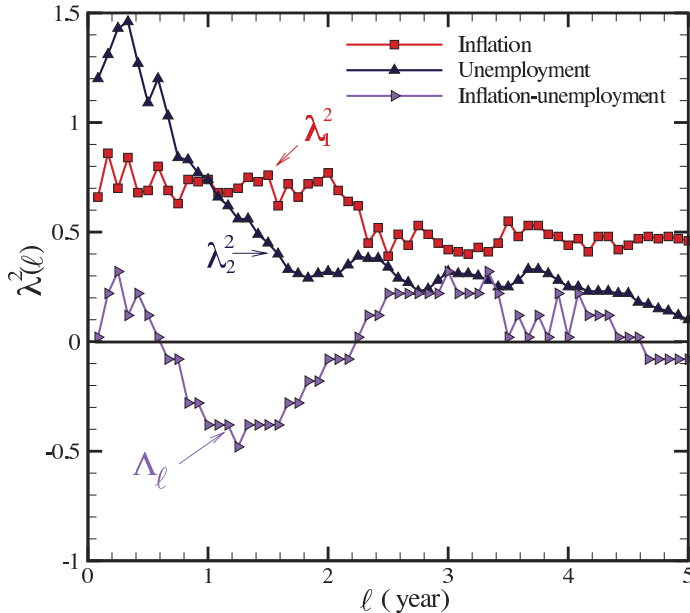


FIG. 4: The non-Gaussian parameter for inflation (square symbol) rates and unemployment (triangle symbol) together with their joint multifractal parameter  $\Lambda_{\ell}$  (right triangle symbol). The periods where  $\Lambda_{\ell}$  is positive means that inflation and unemployment rates are directly proportional to each other. From around month 6 to month 26 where  $\Lambda_{\ell}$  is negative, inflation and unemployment rates are inversely proportional to each other.

#### IV. CONCLUSIONS

In this work, the bivariate multifractal random walk method has been implemented in order to describe the criticality or uncertainty emerged from the coupled multifractality of two systems in the context of the joint log-normal cascade model. As coupling is a scaling concept, therefore we have indicated in this paper that there are

certain scales that possess strong couplings. The parameter under investigation was the joint multifractal parameter,  $\Lambda_{\ell}$ , illustrated at various scales,  $\ell$ . The estimation of  $\Lambda_{\ell}$  revealed its dependence on both the non-Gaussian parameter,  $\lambda_{\diamond}^2(\ell)$ , of each individual process, and the cross correlation of the stochastic variances. It was also shown that the uncertainty of the coupled systems is non-symmetric. This means that the occurrence of large fluctuations in one of the two systems caused by the other system may differ if it were to be from the other way round.

The value of the non-Gaussianity parameter in the context of the present study ( $\lambda_{\diamond}^2(\ell)$ ) decreases with  $\ell$  for both of the series at intermediate scales which is around one month. This behaviour remains just for gold data while for oil we found that after passing the intermediate scale, an increase for  $\lambda_{\text{oil}}^2(\ell)$  takes place. As indicated in Fig. 2 (a), our results confirmed that there exists a powerful coupling together with a high uncertainty between oil and gold markets around a one month time-scale in a situation that the gold market is tending to a normal state ( $\lambda_{\text{gold}}^2 \rightarrow 0$ ). In order to show which market causes large fluctuations in the other market, variance of the conditional distribution of the stochastic local magnitudes of each data set have been evaluated. Our results corroborated that the large fluctuations in the oil market give rise to the occurrence of large fluctuations in the gold market forming a high conditional variance of gold|oil. However, this is not established for the conditional variance of oil|gold, as indicated in Fig. 3 (a), where the conditional variance of gold|oil is greater than the conditional variance of oil|gold. The conditional cross correlation (Eq. (18)) of gold|oil is greater than for oil|gold at all time scale (Fig. 3 (b,c,d)). These results confirmed that the oil market imposes large fluctuations on the gold market and it is consistent with the results given by the conditional variance (Eq. (16)).

The results obtained here confirm that when the systems are in their normal situations and the cross correlations of large fluctuations are long-range, the joint multifractality,  $\Lambda_{\ell}$ , is strongest. This level was hereby chosen to be called the resonance state. In the case where at least one of the systems is in a high uncertainty state,  $\Lambda_{\ell}$  decreases. This issue is caused by the weak cross correlations between the systems. An important finding of this article is deduced from Eq. (15) which shows that in order to control the uncertainty in a coupled system, creating an uncertainty state in both of the two individual systems in the presence of their cross correlation proves adequate for a least uncertainty in the coupled system. This is a sign of reciprocal or in other words depending fluctuations between two systems. This would result in great joint changes in the two systems involved.

The proposed method has been applied to two financial time series. The conclusion is that there exists a robust coupling of the oil and gold markets around one month time-scale in the presence of long-range cross correlations between the two markets. However before and after the



one month time scale the systems behave differently. In a sense that the reduction of coupled uncertainty is due to the weak cross correlation of large changes between the two systems. This proves the importance of the resonance region which for the gold and oil markets is about one month. This may provide opportunity for organizations that experience loss due to oil or gold fluctuations

to be able to plan ahead for preventing loss or gaining benefit.

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- [1] N. Marwan, J. Kurths. *Phys. Lett. A* **302**, 299 (2002).
  - [2] S. Frenzel, B. Pompe, *Phys. Rev. Lett.* **99**, 204101 (2007).
  - [3] Jie Sun, Carlo Cafaro and Erik M. Boltt, *Entropy* **16**, 3416-3433 (2014).
  - [4] G. Jafari, A.H. Shirazi, A. Namaki, R. Raei, *Computing in Science and Engineering* **13** (6), 84-89 (2011).
  - [5] S. Mehraban, A. Shirazi, M. Zamani, G. Jafari, *EPL* **103**, 50011 (2013).
  - [6] A.H. Shirazi, G.R. Jafari, J. Davoudi, J. Peinke, M.R.R. Tabar, M. Sahimi, *Journal of Statistical Mechanics*, P07046 (2009).
  - [7] Eugene A. Eliseev, Sergei V. Kalinin, Yijia Gu, Maya D. Glinchuk, Victoria Khist, Albina Borisevich, Venkatraman Gopalan, Long-Qing Chen, and Anna N. Morozovska, *Phys. Rev. B* **88**, 224105 (2013).
  - [8] Tayeb Jamali, S. Vasheghani Farahani, Mona Jannesar, George Palasantzas, G.R. Jafari, *J. Appl. Phys.* **117**, 175308 (2015).
  - [9] M.H. Wang, N. Chen and J.H. Wang, *Brain Res.* **6**, 1550 (2014).
  - [10] B.B. Mandelbrot, *Fractal and Scaling in Finance: Discontinuity, Concentration, Risk*, Springer-Verlag, New York (1997).
  - [11] B.B. Mandelbrot, A. Fisher, L. Calvet, *Cowles Foundation Discussion Paper*, 1165 (1997).
  - [12] E. Bacry, J. Delour, J. F. Muzy, *Physica A* **299**, 84-92 (2001).
  - [13] E. Bacry, J. Delour, J. F. Muzy, *Phys. Rev. E* **64**, 026103 (2001).
  - [14] L. Hedayatifar, M. Vahabi, G.R. Jafari, *Phys. Rev. E* **84**, 021138 (2011).
  - [15] B. Castaing, Y. Gagne, E. Hopfinger, *Physica D* **46**, 177 (1990).
  - [16] B. Chabaud, A. Naert, J. Peinke, F. Chilla, B. Castaing, B. Hebral, *Phys. Rev. Lett.* **73**, 3227 (1994).
  - [17] J. F. Muzy, J. Delour and E. Bacry, *Eur. Phys. J. B* **17**, 537 (2000).
  - [18] E. A. Novikov, *Phys. Fluids A* **2** (1990) 814; *Phys. Rev. E* **50**, R3303 (1994).
  - [19] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, England, 1995).
  - [20] L. Sorriso-Valvo, V. Carbone, P. Veltri, G. Consolini, R. Bruno, *Geophys. Res. Lett.* **26** 13 (1999).
  - [21] Luciano Telesca, Michele Lovallo, *J. Stat. Mech.* P07001 (2011).
  - [22] S. Hajian and M. Sadegh Movahed, *Physica A* **389**, 4942-4957 (2010).
  - [23] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, Y. Dodge, *Nature (London)* **381**, 767 (1996).
  - [24] K. Kiyono, Z. R. Struzik, Y. Yamamoto, *Phys. Rev. Lett.* **96**, 068701 (2006).
  - [25] G. R. Jafari, M. Sadegh Movahed, P. Noroozadeh, A. Bahraminasab, M. Sahimi, F. Ghasemi, M. Reza Rahimi Tabar, *International Journal of Modern Physics C* **18**, 1689 (2007).
  - [26] K. Kiyono, Z. R. Struzik, N. Aoyagi, S. Sakata, J. Hayano, Y. Yamamoto, *Phys. Rev. Lett.* **93** 178103 (2004).
  - [27] K. Kiyono, Z. R. Struzik, N. Aoyagi, F. Togo, Y. Yamamoto, *Phys. Rev. Lett.* **95**, 058101 (2005).
  - [28] H. Dashtian, G.R. Jafari, Z.K. Lai, M. Masihi, M. Sahimi, *Transport in porous media* **90**, 445-464 (2011).
  - [29] P. Manshour, S. Saberi, M. Sahimi, J. Peinke, A. F. Pacheco, M.R. Rahimi Tabar, *Phys. Rev. Lett.* **102**, 014101 (2009).
  - [30] S. Shadhkoo, G.R. Jafari, *The European Physical Journal B* **72**, 679-683 (2009).
  - [31] Lciano Telesca, Ashutosh Chamoli, Michele Lovallo, Tony Alfredo Stabile, *Pure and Applied Geophysics* 10.1007/s00024-014-0862-3 (2014).
  - [32] S.K. Aggarwal, M. Lovallo, P. P. Khan, B. K. Rastogi, L. Telesca, *Physica A* **426**, 56-62 (2015).
  - [33] F. Shayeganfar, S. Jabbari-Farouji, M. S. Movahed, G. R. Jafari, M. R. R. Tabar, *Phys. Rev. E* **80**, 061126 (2009).
  - [34] F. Shayeganfar, S. Jabbari-Farouji, M. S. Movahed, G. R. Jafari, M. R. R. Tabar, *Phys. Rev. E* **81** 061404 (2010).
  - [35] F. Shayeganfar, M.S. Movahed, G.R. Jafari, *Chemical Physics* **423**, 167-172 (2013).
  - [36] A. Arneodo, J. F. Muzy, D. Sornette, *Eur. Phys. J. B* **2**, 277 (1998).
  - [37] Xiao-Hui Ni, Zhi-Qiang Jiang, Wei-Xing Zhou, *Phys. Lett. A* **373**, 3822 (2009).
  - [38] O. Løvstletten and M. Rypdal, *Phys. Rev. E* **85**, 046705 (2012).
  - [39] C. Ludeña and P. Soulier, *Bernoulli* **20**, 334 (2014).
  - [40] A. Fauth, C. A. Tudor, *IEEE Transactions on Information Theory* **60**, 1963 (2014).
  - [41] J. Ludescher, A. Bunde, *Phys. Rev. E* **90**, 062809 (2014).
  - [42] M.-E. Brachet, E. Taflin, J.-M. Tcheou, *Chaos, Solitons and Fractals* **11**, 2343 (2000).
  - [43] D. Sornette, *Critical phenomena in natural sciences: chaos, fractals, selforganization and disorder: concepts and tools* (2nd ed.) Springer (2009).
  - [44] A. Arneodo, S. Roux, J. F. Muzy, *J. Phys. II France* **7**, 363 (1997).
  - [45] J. W. Kantelhardt, S. A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde and H. E. Stanley, *Physica A*, **316**, 87 (2002).
  - [46] Danijel Grahovac, Nikolai N. Leonenko, *Chaos Solitons Fractals* **65** 78 (2014).
  - [47] J. F. Muzy, D. Sornette, J. Delour, A. Arneodo, *Quantitative Finance*, **1**, 131 (2001).
  - [48] H. Markowitz, *Portfolio selection: Efficient diversification of investments* (John Wiley and Sons, New-York,



- 1959).
- [49] Udo von Toussaint, Rev. Mod. Phys. 83, 943 (2011).
  - [50] G. D' Agostini, Rept.Prog.Phys. 66 ,1383-1420 (2003).
  - [51] D.S.Sivia and J.Skilling, *Data Analysis: A Bayesian Tutorial*, Oxford University Press (2006).
  - [52] A. Arneodo, E. Bacry, S. Manneville, J. F. Muzy, Phys. Rev. Lett. **80**, 708 (1998).
  - [53] [goldratefortoday.org](http://goldratefortoday.org)
  - [54] A. W. Phillips, *Economica* **25**, 283-299 (1958).
  - [55] <http://www.usinflationcalculator.com/>